STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

R 122

# On the Coefficients of the Modular Invariant $J(\tau)$

A. van Wijngaarden

1953

KONINKLIJKE NEDERLANDSE AKADEMIE
VAN WETENSCHAPPEN

# On the coefficients of the modular invariant $J(\tau)$

BY

A. VAN WIJNGAARDEN

(REPORT R 122 OF THE COMPUTATION DEPARTMENT OF THE
MATHEMATICAL CENTRE, AMSTERDAM)

Reprinted from Proceedings, Series A, Vol. LVI, No. 4, 1953

1953
NORTH-HOLLAND PUBLISHING COMPANY
(N.V. NOORD-HOLLANDSCHE UITGEVERS MAATSCHAPPIJ)
AMSTERDAM

MATHEMATICS

# ON THE COEFFICIENTS OF THE MODULAR INVARIANT $J(\tau)$

BY

A. VAN WIJNGAARDEN

(*Report R 122 of the Computation Department of the Mathematical Centre, Amsterdam*)

(Communicated by Prof. B. VAN DER POL at the meeting of September 26, 1953)

## 1. *Introduction*

In recent years several authors paid attention to properties and the computation of the Fourier coefficients of the modular invariant $J(\tau)$. H. S. ZUCKERMAN [9], showed a relationship with the partition function, which enabled him to compute the first 24 coefficients with relative ease, thus extending considerably the first tabulation by W. E. H. BERWICK [1], who gave the first 7 coefficients. In order to proceed in this way one needs, however, first tables of the partition function that go further than the existing ones of H. GUPTA [2], [3]. D. H. LEHMER [4] proved several congruences and relations between the coefficients themselves and between the coefficients, the divisor functions and Ramanujan's tau function, and computed, moreover, the 25-th coefficient. B. VAN DER POL [7] gave also many relations of that last type, and relations between $J(\tau)$ and the theta functions. In his paper also values of the next three coefficients are given that were computed by the author of this paper, the last of which being erroneous due to a regrettable coincidence. In the present paper a table of the first 100 coefficients is given, together with some new properties, found empirically and proved afterwards.

The author is indebted to professor VAN DER POL for many substantial contributions and to Miss G. BOTTERWEG, Miss H. C. HAGENAAR and Mr W. KLEIN for their share in the numerical work.

## 2. *Relations between theta functions and* $J(\tau)$

Let $M_k$ be a function of the integer $k$ and the three numbers $x$, $y$ and $z$ defined by

$$M_k = x^k + y^k + z^k, \tag{2.1}$$

so that always $M_0 = 3$, and let, moreover, $x$, $y$ and $z$ be such that

$$M_1 = 0. \tag{2.2}$$

Then we have

$$M_2 = x^2 + y^2 + z^2 = -2(xy + yz + zx),$$
$$M_3 = x^3 + y^3 + z^3 = 3xyz$$

so that

$$^1/_3\, M_3\, M_k = xyz\,(x^k + y^k + z^k),$$
$$^1/_2\, M_2\, M_{k+1} = -\,(xy + yz + zx)\,(x^{k+1} + y^{k+1} + z^{k+1}),$$

from which it follows by addition, and taking into account (2.1) and (2.2)

$$M_{k+3} = {}^1/_2\, M_2\, M_{k+1} + {}^1/_3\, M_3\, M_k. \tag{2.3}$$

The solution of this recurrence relation that satisfies (2.2) can be written down in explicit form.

Introducing

$$I = ({}^1/_2\, M_2)^3\, ({}^1/_3\, M_3)^{-2} \tag{2.4}$$

one finds after some tedious arithmetic

$$\begin{cases} M_0 = 3, \qquad M_1 = 0, \\ M_{2k} = ({}^1/_2\, M_2)^k \sum\limits_{h=0}^{[k/3]} \dfrac{2k}{k-h} \binom{k-h}{2h} I^{-h}, \qquad k > 0, \\ M_{2k+1} = ({}^1/_2\, M_2)^{k+1/2}\, I^{-1/2} \sum\limits_{h=0}^{[k-1/3]} \dfrac{2k+1}{k-h} \binom{k-h}{2h+1} I^{-h}, \qquad k > 0, \\ M_{-k} = ({}^1/_2 M_2)^{-1/2k} \sum\limits_{h=0}^{[k/3]} (-)^h \dfrac{k}{k-2h} \binom{k-2h}{h} I^{1/2k-h}, \qquad k > 0. \end{cases} \tag{2.5}$$

Inspecting the exponents of $M_2$ and $I$ in these formulae it appears that the product $M_k\, M_{-k}$ takes a particularly simple form. In effect

$$\begin{cases} M_0\, M_0 = 9, \qquad M_1\, M_{-1} = 0, \\ M_{2k}\, M_{-2k} = I^{k-[k/3]-[2k/3]} \sum\limits_{h=0}^{[k/3]} \dfrac{2k}{k-h} \binom{k-h}{2h} I^{[k/3]-h} \\ \qquad\qquad \cdot \sum\limits_{h=0}^{[2k/3]} (-)^h \dfrac{2k}{2k-2h} \binom{2k-2h}{h} I^{[2k/3]-h}, \qquad k > 0. \\ M_{2k+1}\, M_{-(2k+1)} = I^{k-[k-1/3]-[2k+1/3]} \sum\limits_{h=0}^{[k-1/3]} \dfrac{2k+1}{k-h} \binom{k-h}{2h+1} I^{[k-1/3]-h} \\ \qquad\qquad \cdot \sum\limits_{h=0}^{[2k+1/3]} (-)^{h+1} \dfrac{2k+1}{2k+1-2h} \binom{2k+1-2h}{h} I^{[2k+1/3]-h}, \qquad k > 0. \end{cases} \tag{2.6}$$

Hence, $M_k M_{-k}$ is a polynomial in $I$ of degree $[k/2]$ with integer coefficients, that for $I = 0$ takes the value 9 if $k \equiv 0 \pmod 3$ and vanishes if $k \not\equiv 0 \pmod 3$. More specifically

$$M_k\, M_{-k} = \sum_{n=0}^{[k/2]} m_k(n)\, I^n, \tag{2.7}$$

where the coefficients $m_k(n)$ for $k = 0\ (1)13$ are given in the following table:

| $k$ | $m_k(0)$ | $m_k(1)$ | $m_k(2)$ | $m_k(3)$ | $m_k(4)$ | $m_k(5)$ | $m_k(6)$ |
|---|---|---|---|---|---|---|---|
| 0 | 9 | | | | | | |
| 1 | 0 | | | | | | |
| 2 | 0 | 2 | | | | | |
| 3 | 9 | − 3 | | | | | |
| 4 | 0 | − 8 | 2 | | | | |
| 5 | 0 | 25 | − 5 | | | | |
| 6 | 9 | − 12 | − 9 | 2 | | | |
| 7 | 0 | − 49 | 49 | − 7 | | | |
| 8 | 0 | 96 | − 40 | − 8 | 2 | | |
| 9 | 9 | − 27 | − 135 | 78 | − 9 | | |
| 10 | 0 | − 150 | 355 | − 100 | − 5 | 2 | |
| 11 | 0 | 242 | − 121 | − 242 | 110 | − 11 | |
| 12 | 9 | − 48 | − 828 | 892 | − 201 | − 0 | 2 |
| 13 | 0 | − 338 | 1521 | − 507 | − 338 | 143 | − 13 |

Now, introducing the Jacobian theta functions [1])

$$\theta_1' = \theta_1'(q) = \{\partial/\partial z\, \theta_1(z, q)\}_{z=0},\ \theta_j = \theta_j(q) = \theta_j(0, q)\ \ (j = 2, 3, 4),$$

it is seen from the first of the two fundamental relations

$$\theta_2^4 - \theta_3^4 + \theta_4^4 = 0, \qquad \theta_1' = \theta_2\, \theta_3\, \theta_4, \tag{2.8}$$

that a set of $x$, $y$ and $z$ satisfying (2.2) is given by $x = \theta_2^4$, $y = -\theta_3^4$, $z = \theta_4^4$.

Accordingly

$$M_k = \theta_2^{4k} + (-\theta_3^4)^k + \theta_4^{4k}, \tag{2.9}$$

and, (confer [7]),

$$I = 2^{-2}\, 3^3\, J(\tau). \tag{2.10}$$

Hence, the equations (2.6) or (2.7) are as many relations between theta functions and the absolute modular invariant $J(\tau)$. Most of these results can be found in [7]. They are given here because of the explicit form into which they have been brought, and, moreover, because slight errors in the values of some coefficients occur in formula (28) of [7], however without any further consequences.

## 3. *Relations between theta functions and the generating functions of the partition functions*

Some classical results are needed for our purpose. Be $p(n)$ the number of partitions of the natural number $n$ into the sum of natural numbers without restrictions. If restrictions are imposed it is denoted by an index.

---

[1]) Here, and in what follows, $q$ is considered as argument rather than $\tau$ ($q = e^{\pi i \tau}$), since only arithmetical properties are examined. The argument is, moreover, suppressed everywhere when it is $q$, but only used in cases like $\theta_4(q^4)$, where others might prefer $\theta_4(0,4\tau)$. It seemed too hard to be consistent and to denote by $J$ or $J(q)$ rather than by $J(\tau)$ a function that owes its name to its modular properties.

The following scheme is proposed. Restrictions on the character of the terms are denoted by a lower index, restrictions on the number of terms by an upper index. Moreover, $e$ stands for *even*, $o$ for *odd*, $d$ for *different*, $s$ for *same*, $p$ for *prime*. A number, if need be together with algebraical symbols, has an obvious meaning, and other symbols may be introduced by additional conventions. Hence, $p_d(n)$ is the number of partitions of $n$ into different (unequal) terms; $p_{do}(n)$ is the number of partitions of $n$ into unequal odd terms; $p_d^o(n)$ is the number of partitions of $n$ into an odd number of unequal terms; $p_{d>6}^3(n)$ is the number of partitions of $n$ into three unequal terms each greater than six; $p_p^2(2n) > 0$, $n > 1$, expresses the conjecture of GOLDBACH, and so on. By definition, a value is attributed to a partition function of the argument 0, viz. the one that suits its generating function.

Some generating functions are in classical notation

$$(3.1)\qquad \begin{cases} q_0 = q_0(q) = \prod_{n=1}^{\infty}(1-q^{2n}) = \sum_{n=-\infty}^{\infty}(-)^n q^{n(3n+1)}, \\ q_1 = q_1(q) = \prod_{n=1}^{\infty}(1+q^{2n}) = \sum_{n=0}^{\infty} p_d(n)\, q^{2n}, \\ q_2 = q_2(q) = \prod_{n=1}^{\infty}(1+q^{2n-1}) = \sum_{n=0}^{\infty} p_{do}(n)\, q^{n}, \\ q_3 = q_3(q) = \prod_{n=1}^{\infty}(1-q^{2n-1}) = \sum_{n=0}^{\infty}(-)^n p_{do}(n)\, q^{n}, \end{cases}$$

where apparently by definition $p_d(0) = p_{do}(0) = 1$.

From the product expansions of the theta functions it follows that

$$(3.2)\qquad \begin{cases} \theta_1' = 2q^{1/4} q_0^3, & \theta_2 = 2q^{1/4} q_0 q_1^2, \\ \theta_3 = q_0 q_2^2, & \theta_4 = q_0 q_3^2. \end{cases}$$

Again, the functions of section 2 take the following form

$$(3.3)\qquad \begin{cases} M_2 = 2q_0^8 q_1^{-8}(2^8 q^2 q_1^{24} + 1), & M_3 = -2^4 3\, q\, q_0^{12}, \\ I = 2^{-8} q^{-2} q_1^{-24}(2^8 q^2 q_1^{24} + 1)^3, \end{cases}$$

whereas the relations (2.8) read

$$(3.4)\qquad 2^4 q\, q_1^8 = q_2^8 - q_3^8, \qquad q_1 q_2 q_3 = 1.$$

From (3.1) and (3.4) one derives easily

$$(3.5)\qquad \begin{cases} q_0(q^2) = q_0(q) q_1(q) = q_0(iq)\, q_1(iq), \\ q_2(q^2) = q_1^{-1}(iq), \qquad q_3(q^2) = q_1^{-1}(q). \end{cases}$$

Hence, from (3.2) and (3.5) it follows

$$(3.6)\qquad \theta_3(q^2) = q_0(iq)\, q_1^{-1}(iq), \qquad \theta_4(q^2) = q_0(q)\, q_1^{-1}(q).$$

At last a special combination will be examined, viz.

$$\begin{aligned}\theta_2^{-8}(q) + \theta_2^{-8}(iq) &= 2^{-8} q^{-2} q_0^{-8}(q^2) \{q_1^{-8}(q) - q_1^{-8}(iq)\} \\ &= 2^{-8} q^{-2} q_0^{-8}(q^2) \{q_3^8(q^2) - q_2^8(q^2)\} \\ &= -2^{-4} q_0^{-8}(q^2) q_1^8(q^2) = -2^{-4} \theta_4^{-8}(q^4),\end{aligned}$$

whence

$$\theta_4^{-8}(q^4) = 2^4 \{\theta_2^{-8}(q) + \theta_2^{-8}(iq)\}. \tag{3.7}$$

4. *The coefficients of* $J(\tau)$

As well known, the coefficients $c(n)$ in the expansion

$$j(\tau) = 2^8 I = 12^3 J(\tau) = \sum_{n=-1}^{\infty} c(n) q^{2n} \tag{4.1}$$

are integers. There are many ways to compute the coefficients, each of one resulting in rather lengthy calculations if $n$ is not very small. This is rather natural since the coefficients grow very rapidly with $n$, and the digits have to come from somewhere. LEHMER [4] has derived multiplicative relations that for special values of $n$ involve a moderate amount of arithmetical operations but all on large numbers. LEHMER [4] and VAN DER POL [7] have given recurrence relations implying divisor functions and Ramanujan's tau-function. Typical examples are the following ones ([7], formula 54a)

$$\begin{cases} \sum_{k=-1}^{n-1} c(k)\, \tau(n-k) = \frac{720}{691} \{91\, \sigma_{11}(n) + 600\, \tau(n)\}, \\ \sum_{k=-1}^{n-1} k c(k)\, \tau(n-k) = 24\, \sigma_{13}(n). \end{cases} \tag{4.2}$$

Here $\tau(n)$ is Ramanujan's function and $\sigma_{2m+1}(n)$ is the sum of the $(2m+1)$th powers of the divisors of $n$, if $m = 1, 2, 3, \ldots$, whereas by definition $\sigma_{2m-1}(0) = \frac{1}{2}\zeta(1-2m) = -B_{2m}/(4m)$ and $B_{2m}$ are the Bernoullian numbers, $B_0 = 1$, $B_2 = 1/6$, $B_4 = -1/30$ .... The $\tau$-function is well tabulated and the $\sigma$-functions are easily computed. Moreover, they are not very large. From both formulae $c(n)$ is found by repeating the same procedure. The second formula has moreover an advantage over the first one in that it yields $(n-1)c(n-1)$, whereas the first one yields $c(n-1)$ directly $(\tau(1) = 1)$, so that in the second case one has an important numerical check in the remainderless division by $n-1$.

Yet, another type of formulae exists, one of which will be investigated now. The third formula (2.7) together with (4.1) yields

$$j(\tau) = 2^7 M_2 M_{-2}. \tag{4.3}$$

Now, $M_2$ is a very simple function. In effect (confer [7])

$$M_2 = 480 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}, \tag{4.4}$$

where it should be remembered that $\sigma_3(0) = 1/240$. Be analogously

$$M_{-2} = \theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8} = 2^{-8} \sum_{n=-1}^{\infty} d(n)\, q^{2n}, \tag{4.5}$$

where all $d(n)$ are positive integers ($n \geqslant -1$). Also,

$$\theta_4^{-8} = 2^4 \sum_{n=0}^{\infty} s(n)\, q^n, \tag{4.6}$$

where $s(0) = 2^{-4}$ and $s(n)$ is a positive integer for $n > 0$. Since $\theta_4(q) = \theta_3(-q)$, also

$$\theta_3^{-8} = 2^4 \sum_{n=0}^{\infty} (-)^n s(n)\, q^n, \tag{4.7}$$

Similarly

$$\theta_2^{-8} = -2^{-8} \sum_{n=-1}^{\infty} t(n)\, q^{2n}, \tag{4.8}$$

where all $t(n)$ are integers ($n \geqslant -1$) with alternating signs. Hence

$$d(n) = 2^{13} s(2n) - t(n), \qquad n \geqslant -1, \tag{4.8}$$

where, of course, $s(-2) = 0$.

The coefficients $s(n)$ and $t(n)$ can be found from the expansions

$$\theta_2 = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)}, \qquad \theta_4 = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2}, \tag{4.9}$$

by applying the following well known artifice. Be $a(n)$ given coefficients for $n \geqslant 0$, and $a(0) \neq 0$. Then the coefficients $b(n)$ satisfying

$$\sum_{n=0}^{\infty} b(n)\, x^n = \{\sum_{n=0}^{\infty} a(n)\, x^n\}^{m-1}$$

can be found by differentiating logarithmically, multiplying by the arising denominators and equating the coefficients of equal powers of $x$. Then one gets the recurrence relation

$$n\, a(0)\, b(n) = \sum_{k=1}^{n} (mk - n)\, a(k)\, b(n-k), \quad n > 0, \quad b(0) = a(0)^{m-1}.$$

This device is particularly useful if the $a(n)$ form a lacunary sequence. In this way one obtains the recurrence relations

$$\begin{cases} n\, s(n) = -2 \sum_{k=1} (-)^k (n + 7k^2)\, s(n-k^2), \quad n > 0, \quad s(0) = 2^{-4}, \\ (n+1)\, t(n) = -\sum_{k=1} \left\{n + 1 + 7\, \frac{k(k+1)}{2}\right\} t\left\{n - \frac{k(k+1)}{2}\right\}, \quad n > -1, \; t(-1) = -1, \end{cases} \tag{4.10}$$

where $k$ takes such values that $n - k^2 \geqslant 0$ resp. $n - k(k+1)/2 \geqslant -1$.

Both formulae allow numerical checks by the division by $n$, resp. $n+1$.

Having obtained $s(n)$ and $t(n)$, $d(n)$ follows from (4.8) and hence $c(n)$ from (4.3), (4.4) and (4.5)

$$c(n) = 240 \sum_{k=0}^{n+1} \sigma_3(k)\, d(n-k) = d(n) + 240 \sum_{k=1}^{n+1} \sigma_3(k)\, d(n-k). \tag{4.11}$$

The final summation is therefore rather simple, all terms being positive and $\sigma_3(k)$ being only a small factor. Of course, first $s(n)$, $t(n)$ and $d(n)$ have to be computed but these functions are of some interest in themselves. Moreover it will be shown that $s$ can be expressed in $t$. To that end one combines (3.7) with (4.6) and (4.8):

$$\theta_4^{-8}(q^4) = 2^4 \sum_{n=0}^{\infty} s(n)\, q^{4n},$$

$$\theta_2^{-8}(q) + \theta_2^{-8}(iq) = -2^{-8}\{\sum_{n=-1}^{\infty} t(n)\, q^{2n} + \sum_{n=-1}^{\infty} (-)^n t(n)\, q^{2n}\} =$$

$$= -2^{-8} \sum_{n=0}^{\infty} t(2n)\, q^{4n},$$

whence

$$s(n) = 2^{-8}\, t(2n), \qquad n \geqslant 0, \tag{4.12}$$

or together with (4.8)

$$d(n) = 2^5\, t(4n) - t(n), \tag{4.13}$$

so that the computation of $c(n)$ may be completely based on that of $t(n)$ and $\sigma_3(n)$. In actual practice, if one wants to compute $c(n)$ up to a fixed upper value of $n$, $N$ say, it is easier to compute $t(n)$ for $-1 \leqslant n \leqslant N$ and $s(n)$ for $N/2 \leqslant n \leqslant 2N$ directly from (4.10) than to compute $t(n)$ for $-1 \leqslant n \leqslant 4N$ and then to apply (4.12).

Actually (4.12) was found empirically. From the conjecture that it should hold followed then (3.7) which was proved afterwards in the given way.

This method of computing $c(n)$ has the disadvantage, however, that after the formation of $s(n)$ and $t(n)$ no internal checks occur in the formulae. It is therefore advisable to have an efficient numerical check on the values of $c(n)$ for all values of $n$.

## 5. *Congruences satisfied by* $c(n)$

One knows many congruences satisfied by the coefficients $c(n)$. LEHMER [4] proved (more than) that

$$(n+1)\, c(n) \equiv 0 \pmod{24}, \tag{5.1}$$

$$c(n) \equiv 0 \pmod 5 \text{ if } n \equiv \pm 2 \pmod 5. \tag{5.2}$$

J. LEHNER [5], [6] proved that

$$\begin{cases} c(n) \equiv 0 \pmod{2^{3\alpha+8}\, 3^{2\beta+3}\, 5^{\gamma+1}\, 7^{\delta}\, 11^{\varepsilon}} \text{ if } 1 < n \equiv 0 \pmod{2^{\alpha}\, 3^{\beta}\, 5^{\gamma}\, 7^{\delta}\, 11^{\varepsilon}}; \\ \qquad\qquad \alpha, \beta, \gamma, \delta \geqslant 1, \quad \varepsilon = 1, 2, 3. \end{cases} \tag{5.3}$$

Moreover he remarked that from the numerical values of $c(n)$ available to him ($n \leqslant 25$) followed that in the case that $n$ is one of the five primes 2, 3, 5, 7 and 11 or a power of one of these primes (thus if $n = 2, 3, 4, 5,$

7, 8, 9, 11, 16, 25), the congruences (5.3) predict the exact power of that prime dividing $c(n)$, $(n \leqslant 25)$.

We shall now prove some other remarkable congruences. From the formulae of section 2 it follows that

$$J(\tau) = {}^1/_6\, M_2^3\, M_3^{-2} = {}^1/_6\,(M_2^3\, M_3)\, M_3^{-3} = 2^2\, 3^{-2}\, M_3^{-3}\, M_9 - 2^2\, 3^{-4}.$$

Combining this with the formulae of section 3 one gets

$$J(\tau) + 2^2\, 3^{-4} = 2^2\, 3^{-2}\, M_3^{-3}\, M_9 = 2^{-10}\, 3^{-5}\, q^{-3}\,(q_2^{72} - q_3^{72} - 2^{36}\, q^9\, q_1^{72}),$$

and after introducing (4.1), with $c(-1) = 1$, $c(0) = 744$,

$$24\, q^3\,\{3q^{-2} + 35.71 + 3 + 3\sum_{n=1}^{\infty} c(n)\, q^{2n}\} = {}^1/_2\,(q_2^{72} - q_3^{72}) - 2^{35}\, q^9\, q_1^{72}.$$

Now, in virtue of (3.1) it holds

$$q_h^{72} = q_h(q)\, q_h^{71}(q) \equiv q_h(q)\, q_h(q^{71}) \qquad (\mathrm{mod}\ 71),\ (h = 1, 2, 3),$$

and moreover $2^{35} \equiv 1 \pmod{71}$, so that

$$q + q^3 + \sum_{n=1}^{\infty} c(n)\, q^{2n+3} \equiv {}^1/_2\,\{q_2(q)\, q_2(q^{71}) - q_3(q)\, q_3(q^{71})\} - q^9\, q_1(q)\, q_1(q^{71}) \qquad (\mathrm{mod}\ 71).$$

Writing, for a moment, $c'(n) = c(n)$ if $n \neq 0$, $c'(0) = 1$, then together with (3.1) again, one gets

$$\sum_{n=1}^{\infty} c'(n)\, q^{2n+3} \equiv {}^1/_2\,\{\sum_{n=0}^{\infty} p_{do}(n)\, q^n \sum_{n=0}^{\infty} p_{do}(n)\, q^{71n} - \sum_{n=0}^{\infty} (-)^n\, p_{do}(n)\, q^n \sum_{n=0}^{\infty} (-)^n\, p_{do}(n)\, q^{71n}\} - q^9 \sum_{n=0}^{\infty} p_d(n)\, q^{2n} \sum_{n=0}^{\infty} p_d(n)\, q^{142n} \qquad (\mathrm{mod}\ 71).$$

By equating the coefficients of the terms containing $q^{2n+3}$ one finds then at last

$$(5.4) \qquad \begin{cases} c(n) \equiv \sum_{k=0}^{[2n+3/71]} p_{do}(k)\, p_{do}(2n+3-71k) - \sum_{k=0}^{[n-3/71]} p_d(k)\, p_d(n-3-71k) \\ \qquad\qquad (\mathrm{mod}\ 71), \quad n \neq 0. \end{cases}$$

The two partition functions $p_d(n)$ and $p_{do}(n)$ have been tabulated by G. N. Watson [8], up to $n = 400$. They are, moreover, easily computable and relatively small numbers. The right hand side of (5.4) contains only very few terms, even less than it appears at first sight since $p_{do}(2) = 0$. For instance, for $n \leqslant 109$, (5.4) runs in full

$$c(n) \equiv p_{do}(2n+3) + p_{do}(2n+3-71) - p_d(n-3) - p_d(n-3-71) \qquad (\mathrm{mod}\ 71), \quad 0 \neq n \leqslant 109.$$

For $n = 100$, for instance, one has

$$c(100) \equiv 354357 + 14157 - 345856 - 165 = 22493 \equiv 57 \pmod{71},$$

which involves really small numbers only in comparison to $c(100)$ itself which is a number of 53 decimal digits.

In the derivation of (5.4) $M_9$ was introduced. If instead one introduces $M_6$ and proceeds along the same lines one gets the analogous congruence

$$(5.5)\qquad \begin{cases} c(n) \equiv \sum_{k=0}^{[2n+2/47]} p_{d_9}(k)\, p_{d_{10}}(2n+2-47k) + \sum_{k=0}^{[n-2/47]} p_d(k)\, p_d(n-2-47k) \\ \qquad\qquad \pmod{47}, \quad n \neq 0. \end{cases}$$

The analogous congruence modulo 23 is more complicated and will be omitted. From the numerical data it appears that similar congruences must exist modulo 41 and modulo 59. Actually the existence of (5.4) and (5.5) was also first found empirically.

## 6. *The computation of* $c(n)$

The coefficients $c(n)$ were compoted for $n = -1\,(1)\,100$ from VAN DER POL'S formula [7, (38)]

$$(6.1)\qquad J(\tau) = {}^{2}/_{27}\,(\theta_2^8 + \theta_3^8 + \theta_4^8)\,(\theta_2^{-8} + \theta_3^{-8} + \theta_4^{-8}),$$

What is the explicit form of (4.3), using the methods described in section 4.

The function $s(n)$ was computed for $n = 0(1)200$ and the function $t(n)$ for $n = -1(1)100$ by means of the selfchecking relations (4.10). They were checked moreover by means of (4.12). From (4.8) and (4.11) followed $d(n)$ and $c(n)$ for $n = -1(1)100$. This whole part was checked by duplication. Next $c(n)$ was computed from the second selfchecking equation (4.2) for $n = 1(1)50$ as an independent check. At last all values of $c(n)$ were submitted to the congruence-checks (5.4) and (5.5). For special values of $n$ the congruences (5.2) and (5.3) were used as checks. The congruence (5.1), although valid for all $n$, was of little use since in both ways of computing the $c(n)$ the factor 24 plays a rôle.

With regard to the remark following (5.3), it is interesting to see whether Lehner's statement also applies to the new powers of primes ($< 13$) that are available now (viz. $n = 27, 32, 49, 64, 81$). This appears to be the case indeed. This phenomenon is rather interesting. In order to have a further check, $c(128)$ was calculated modulo $2^{30}$ from Lehmer's duplication formula [4]. It was found that $c(128) \equiv 2^{29} \pmod{2^{30}}$ so that also in this case Lehner's statement holds.

The values of $c(n)$ are given in the following table.

| $c(n)$ | $n$ |
|---|---|
| 1 | −1 |
| 744 | 0 |
| 1 96884 | 1 |
| 214 93760 | 2 |
| 8642 99970 | 3 |
| 2 02458 56256 | 4 |
| 33 32026 40600 | 5 |
| 425 20233 00096 | 6 |
| 4465 69940 71935 | 7 |
| 40149 08866 56000 | 8 |
| 3 17644 02297 84420 | 9 |
| 22 56739 33095 93600 | 10 |
| 146 21191 14995 19294 | 11 |
| 874 31371 96857 75360 | 12 |
| 4872 01011 17981 42520 | 13 |
| 25497 82738 94105 25184 | 14 |
| 1 26142 91646 57818 43075 | 15 |
| 5 93121 77242 14450 58560 | 16 |
| 26 62842 41315 07752 45160 | 17 |
| 114 59912 78844 47865 13920 | 18 |
| 474 38786 80123 41688 13250 | 19 |
| 1894 49976 24889 33900 28800 | 20 |
| 7318 11377 31813 75192 45696 | 21 |
| 27406 30712 51362 46549 29920 | 22 |
| 99710 41659 93718 26935 33820 | 23 |
| 3 53074 53186 56142 70998 77376 | 24 |
| 12 18832 84330 42251 04333 51500 | 25 |
| 41 07899 60190 30790 91576 38144 | 26 |
| 135 35635 41518 64687 86750 77500 | 27 |
| 436 56892 24858 87663 46104 01280 | 28 |
| 1379 83758 34642 99992 55422 88376 | 29 |
| 4278 07822 44213 26256 70582 27200 | 30 |
| 13023 36938 25770 29512 80448 73221 | 31 |
| 38960 80061 70995 91189 43000 98560 | 32 |
| 1 14632 93989 00810 63777 96110 90240 | 33 |
| 3 31962 77091 39267 16726 36796 06784 | 34 |
| 9 46816 61357 02260 43164 62634 38600 | 35 |
| 26 61436 58257 53796 26887 21518 75584 | 36 |
| 73 77316 99697 25069 76080 17928 54360 | 37 |
| 201 76878 99472 28738 64858 00437 76000 | 38 |
| 544 76388 17516 16630 12316 54104 77688 | 39 |
| 1452 68925 44393 62169 79435 54293 76000 | 40 |
| 3827 76775 17393 63485 06559 83311 30120 | 41 |
| 9970 41660 02174 43268 73940 99688 24320 | 42 |
| 25683 33470 63954 06994 77401 18663 19670 | 43 |
| 65452 36773 14992 68312 17028 36951 44960 | 44 |
| 1 65078 82156 81861 74782 49628 31551 42200 | 45 |
| 4 12189 63080 52167 73489 54445 72343 33696 | 46 |
| 10 19253 51589 15767 91938 65201 10914 37835 | 47 |
| 24 96774 10595 07166 92603 31512 31996 72320 | 48 |
| 60 60574 41541 37209 99542 37822 28126 50932 | 49 |
| 145 81598 45321 50199 97540 39132 61539 84000 | 50 |

| $c(n)$ | | | | | | | | $n$ |
|---|---|---|---|---|---|---|---|---|
| | 347 | 82974 | 25351 | 24906 | 52111 | 11193 | 03264 | 16268 | 51 |
| | 822 | 82309 | 23604 | 86379 | 46346 | 57066 | 92508 | 05760 | 52 |
| | 1930 | 75525 | 46782 | 25741 | 67329 | 52965 | 87752 | 61720 | 53 |
| | 4494 | 97224 | 12333 | 74771 | 55078 | 53776 | 07541 | 22752 | 54 |
| | 10384 | 83010 | 58794 | 97940 | 68925 | 15368 | 59324 | 35825 | 55 |
| | 23814 | 07585 | 30992 | 24134 | 99951 | 81283 | 96335 | 84128 | 56 |
| | 54214 | 49889 | 87656 | 47230 | 00378 | 95797 | 97720 | 88000 | 57 |
| 1 | 22553 | 65475 | 04082 | 06615 | 35516 | 23305 | 01657 | 60000 | 58 |
| 2 | 75134 | 11092 | 85948 | 64606 | 92553 | 08616 | 87146 | 59374 | 59 |
| 6 | 13542 | 89505 | 30361 | 36170 | 69338 | 27228 | 48587 | 77600 | 60 |
| 13 | 59250 | 92428 | 36550 | 38097 | 01809 | 16661 | 62894 | 74168 | 61 |
| 29 | 92109 | 83800 | 07688 | 36650 | 74958 | 85452 | 33318 | 70720 | 62 |
| 65 | 45530 | 43491 | 65030 | 30643 | 85476 | 04156 | 99953 | 65270 | 63 |
| 142 | 31976 | 35972 | 71606 | 23108 | 02114 | 65424 | 36536 | 81152 | 64 |
| 307 | 60954 | 73477 | 19676 | 30396 | 15540 | 12847 | 95239 | 17200 | 65 |
| 661 | 00917 | 73782 | 87162 | 74459 | 09215 | 08064 | 15869 | 54240 | 66 |
| 1412 | 35833 | 72861 | 18490 | 82870 | 80245 | 89187 | 32135 | 44410 | 67 |
| 3001 | 00414 | 97911 | 12962 | 58941 | 10839 | 46623 | 40095 | 18080 | 68 |
| 6341 | 98425 | 35335 | 41630 | 77601 | 14920 | 60361 | 94613 | 13664 | 69 |
| 13331 | 26252 | 93210 | 23532 | 85518 | 96736 | 23687 | 92354 | 81600 | 70 |
| 27877 | 50248 | 90624 | 32847 | 67184 | 93296 | 34876 | 93051 | 98947 | 71 |
| 57998 | 94663 | 06862 | 70977 | 78971 | 24287 | 02702 | 89346 | 56000 | 72 |
| 1 | 20064 | 76859 | 24154 | 07996 | 57067 | 63561 | 79539 | 59481 | 73320 | 73 |
| 2 | 47334 | 29811 | 83106 | 50913 | 62656 | 13239 | 67886 | 40929 | 91488 | 74 |
| 5 | 07071 | 19308 | 98997 | 08057 | 00789 | 06280 | 84219 | 65196 | 46750 | 75 |
| 10 | 34690 | 66408 | 50426 | 35622 | 63168 | 39259 | 82257 | 41159 | 46496 | 76 |
| 21 | 01594 | 58102 | 75143 | 25069 | 10589 | 02482 | 07991 | 00864 | 59520 | 77 |
| 42 | 49352 | 00246 | 86459 | 96896 | 93275 | 41404 | 17894 | 12398 | 69440 | 78 |
| 85 | 53998 | 18184 | 24975 | 89405 | 37694 | 48098 | 79634 | 98086 | 43878 | 79 |
| 171 | 44484 | 30238 | 56632 | 32305 | 05079 | 66626 | 55430 | 46332 | 41600 | 80 |
| 342 | 15552 | 55551 | 89176 | 73198 | 38691 | 23583 | 94201 | 19784 | 93364 | 81 |
| 679 | 98684 | 36672 | 14052 | 17195 | 40980 | 18582 | 52260 | 99449 | 65120 | 82 |
| 1345 | 82384 | 70689 | 81684 | 95259 | 62168 | 82155 | 84589 | 79008 | 27370 | 83 |
| 2652 | 88632 | 13847 | 03560 | 25223 | 21296 | 59440 | 09217 | 23815 | 85408 | 84 |
| 5208 | 62134 | 25202 | 53933 | 69315 | 34883 | 96012 | 72044 | 83857 | 83600 | 85 |
| 10186 | 63549 | 71409 | 56830 | 21681 | 12072 | 29975 | 61148 | 07976 | 01792 | 86 |
| 19845 | 94685 | 77153 | 87241 | 69587 | 80804 | 25504 | 86362 | 87388 | 82125 | 87 |
| 38518 | 94383 | 02834 | 97365 | 36939 | 13362 | 43138 | 88225 | 01457 | 92000 | 88 |
| 74484 | 51892 | 92890 | 17811 | 71998 | 98327 | 68142 | 07693 | 12594 | 10120 | 89 |
| 1 | 43507 | 17246 | 72834 | 53885 | 51522 | 23427 | 82991 | 19235 | 32076 | 03200 | 90 |
| 2 | 75501 | 04261 | 67891 | 53749 | 08061 | 78938 | 36796 | 95113 | 39297 | 83496 | 91 |
| 5 | 27036 | 05805 | 32817 | 64188 | 08922 | 00416 | 29201 | 19197 | 55057 | 56160 | 92 |
| 10 | 04730 | 45344 | 09390 | 42843 | 89896 | 53654 | 12981 | 69030 | 71458 | 27840 | 93 |
| 19 | 08864 | 09832 | 13103 | 02488 | 60473 | 90986 | 18405 | 93893 | 84773 | 79584 | 94 |
| 36 | 14432 | 17930 | 44626 | 81879 | 67680 | 91204 | 64684 | 97513 | 08362 | 05250 | 95 |
| 68 | 21306 | 83268 | 93807 | 76546 | 62982 | 56534 | 65084 | 00341 | 84769 | 04448 | 96 |
| 128 | 31568 | 45093 | 05662 | 37049 | 15719 | 10171 | 04861 | 21743 | 36342 | 89960 | 97 |
| 240 | 60143 | 44493 | 76049 | 97591 | 58609 | 03804 | 73418 | 08640 | 16968 | 39680 | 98 |
| 449 | 72195 | 69801 | 18067 | 40150 | 81827 | 51777 | 54986 | 40947 | 29105 | 49646 | 99 |
| 837 | 98831 | 11070 | 74769 | 12751 | 95038 | 47574 | 52703 | 80191 | 83390 | 72000 | 100 |

## REFERENCES


1. Berwick, W. E. H., An invariant modular equation of the fifth order, Quart. Journ. of Math., **47**, 94–103 (1916).
2. Gupta, H., A table of partitions, Proc. London Math. Soc., **39**, 142–149 (1935).
3. ———, A table of partitions, II, Proc. London Math. Soc., **42**, 546–549 (1937).
4. Lehmer, D. H., Properties of the coefficients of the modular invariant $J(\tau)$, Amer. Journ. of Math., **64**, 488–502 (1942).
5. Lehner, J., Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$, Amer. Journ. of Math., **71**, 136–148 (1949).
6. ———, Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$, Amer. Journ. of Math., **71**, 373–386 (1949).
7. Pol, B. van der, On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications, I, II, Proc. Kon. Ned. Akad. v. Wetensch. Ser. A, **54** = Indagationes Math. **13**, 261–284 (1951).
8. Watson, G. N., Two tables of partitions, Proc. London Math. Soc., **42**, 550–556 (1937).
9. Zuckerman, H. S., The computation of the smaller coefficients of $J(\tau)$, Bull. Amer. Math. Soc., **45**, 917–919 (1939).